

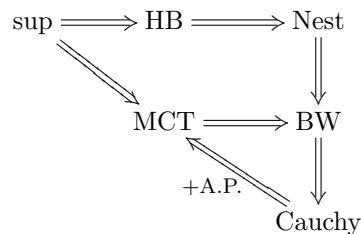
Assignment 3.

This homework is due *Thursday*, September 19.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

1. EXERCISES

- (1) (1.4.36) Show that the Borel σ -algebra \mathcal{B} is the smallest σ -algebra \mathcal{A} that contains all intervals of the form $[a, b)$, where $a < b$. (*Hint*: Show that *both* \mathcal{B} and \mathcal{A} contain *both* open intervals and intervals of the form $[a, b)$.)
- (2) *There used to be a problem here, but a UFO came and abducted it. Disregard this item.*
- (3) (a) (1.4.33) Show that the Nested Set Theorem is false for closed but possibly unbounded sets. (That is, give an example of a nested closed set family for which the conclusion of the Nested Set Theorem fails.)
 (b) Let $(a_1, b_1) \supseteq (a_2, b_2) \supseteq \dots$ be a nested family of open intervals. Show that they may not have a common point.
- (4) Show that the Heine–Borel theorem is false for:
 - (a) open covers of an open bounded set. (That is, give an example of an open bounded set and its open cover for which the conclusion of the Heine–Borel theorem fails.)
 - (b) open covers of a closed unbounded set.
- (5) In lectures, the following implications were proved or at least sketched (sup = Completeness Axiom, HB = Heine–Borel Theorem, Nest = Nested Set Theorem, BW = Bolzano–Weierstrass Theorem, MCT = Monotone Convergence Theorem, A.P. = Archimedean Principle):



Prove enough implications to make the top five statements equivalent to each other.

- (6) A real number c is called a *cluster point* of a sequence $\{a_n\}$ if a subsequence of $\{a_n\}$ converges to c . Show that the set of all cluster points of a sequence in \mathbb{R} is a closed set.
- (7) (\sim 1.5.40) Prove that a bounded sequence in \mathbb{R} converges to a number $a \in \mathbb{R}$ if and only if the set of cluster points of this sequence is the singleton $\{a\}$.

2. EXTRA EXERCISE

- (8) Prove or disprove. For any closed set $F \subseteq \mathbb{R}$, there is a sequence in \mathbb{R} whose set of cluster points is precisely F .