Assignment 3.

This homework is due *Thursday*, September 19.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

1. Exercises

(1) (1.4.36) Show that the Borel σ-algebra $\mathcal{B}$ is the smallest σ-algebra $\mathcal{A}$ that contains all intervals of the form $[a, b)$, where $a < b$. (*Hint:* Show that *both* $\mathcal{B}$ and $\mathcal{A}$ contain *both* open intervals and intervals of the form $[a, b)$.)

(2) *There used to be a problem here, but a UFO came and abducted it. Disregard this item.*

(3) (a) (1.4.33) Show that the Nested Set Theorem is false for closed but possibly unbounded sets. (That is, give an example of a nested closed set family for which the conclusion of the Nested Set Theorem fails.)

(b) Let $(a_1, b_1) \supseteq (a_2, b_2) \supseteq \dots$ be a nested family of open intervals. Show that they may not have a common point.

(4) Show that the Heine–Borel theorem is false for:

(a) open covers of an open bounded set. (That is, give an example of an open bounded set and its open cover for which the conclusion of the Heine–Borel theorem fails.)

(b) open covers of a closed unbounded set.

(5) In lectures, the following implications were proved or at least sketched (sup = Completeness Axiom, HB = Heine–Borel Theorem, Nest = Nested Set Theorem, BW = Bolzano–Weierstrass Theorem, MCT = Monotone Convergence Theorem, A.P. = Archimedean Principle):

$$\begin{tikzcd}
\text{sup} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{HB} \arrow[r, Rightarrow] & \text{Nest} \arrow[d, Rightarrow] \\
 & \text{MCT} \arrow[r, Rightarrow] & \text{BW} \arrow[d, Rightarrow] \\
 & & \text{Cauchy} \arrow[ul, Rightarrow, "+\text{A.P.}"]
\end{tikzcd}$$

Prove enough implications to make the top five statements equivalent to each other.

(6) A real number c is called a *cluster point* of a sequence $\{a_n\}$ if a subsequence of $\{a_n\}$ converges to c. Show that the set of all cluster points of a sequence in $\mathbb{R}$ is a closed set.

(7) ($\sim$1.5.40) Prove that a bounded sequence in $\mathbb{R}$ converges to a number $a \in \mathbb{R}$ if and only if the set of cluster points of this sequence is the singleton $\{a\}$.

2. Extra exercise

(8) Prove or disprove. For any closed set $F \subseteq \mathbb{R}$, there is a sequence in $\mathbb{R}$ whose set of cluster points is precisely F.